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Generalization of Zero Bounds and Zero-Free Regions for Quaternionic Polynomials

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Abstract— In this paper, we present certain results on the zero bounds and zero free regions for quaternionic polynomials by relaxing the condition of monotonicity on the coefficients of a polynomial and thereby obtain generalizations and refinements of many known results.

Index Terms— Enestrom-Kakeya Theorem, Quaternionic Polynomial, Regular functions, Zeros.

I. INTRODUCTION

If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n. Then Enestrom-Kakya [6] proved the following result.

Theorem A: If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that $a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0$, then $P(z)$ has all its zeros in $|z| \leq 1$. Later on Joyall et al. [5] extended Theorem A by relaxing the condition of non-negativity proved the following result.

Theorem B: If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that $a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0$, then $P(z)$ has all its zeros in $|z| \leq \frac{|a_n| - a_0 + |a_0|}{|a_n|}$.

The above results were generalized by Shah [7] by proving the following more general form of Enestrom-Kakya Theorem.

Theorem C: If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that for some positive integer p, $a_p \geq a_{p-1} \geq \ldots \geq a_1 \geq a_0$, then P(z) has all its zeros in

$$|z| \leq \frac{M_p + a_p - a_0 + |a_0|}{|a_n|}.$$

Where $M_p = \sum_{j=p+1}^{n} |a_j - a_{j-1}|.$

II. BACKGROUND

Quaternions, discovered by the mathematician William Rowan Hamilton in the 19th century, extend the concept of complex numbers. While complex numbers consist of a real part and an imaginary part, quaternions incorporate three imaginary components, thus forming a four-dimensional number system. Quaternionic polynomials are polynomials whose coefficients and variables belong to this quaternionic number system. These numbers are generally represented in the form $q=\alpha+\beta i+\gamma j+\delta k$ where $\alpha,\beta,\gamma,\delta \in R$ and i,j,k are fundamental quarternion units satisfy the multiplication rules $i^2=j^2=k^2=ijk=-1$. The set of all quaternions is denoted by H in honour of Sir Hamilton .Multiplication of quarternions is not commutative in general but H is at least division ring and also forms a four dimensional vector space over R with $\{1,i,j,k\}$ as a basis. Let $P_n=\{p, p(q)=\sum_{l=0}^{n} q^l a_l, q\in H\}$ denote the nth-degree polynomials with quarternionic variable $q \in H$ and $a_l, 0 \leq l \leq n$ are either real or quarternion. Carney et al. [1] proved the extension of Enestrom-Kakya Theorem for quaternionic polynomials in the form of following results.

Theorem D: All the zeros of the polynomial $p \in P_n$ of degree n with real coefficients such that $a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0$ lie in $|q| \leq 1$.

Theorem E: If $p(q)=\sum_{l=0}^{n} q^l a_l$ is a polynomial of degree n with quaternionic coefficients and quaternionic variable where $a_l=\alpha+\beta i+\gamma j+\delta k, 0 \leq l \leq n$ and satisfying

$$\alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_1 \geq \alpha_0, \beta_n \geq \beta_{n-1} \geq \ldots \geq \beta_1 \geq \beta_0,$$
$$\gamma_n \geq \gamma_{n-1} \geq \ldots \geq \gamma_1 \geq \gamma_0, \delta_n \geq \delta_{n-1} \geq \ldots \geq \delta_1 \geq \delta_0,$$

Then all the zeros of $p(q)$ lie in

$$|q| \leq \frac{1}{|a_n|}\left[U_\alpha + U_\beta + U_\gamma + U_\delta\right],$$

Where

$$U_t = |t_0| - t_0 + t_n.$$

In this paper, we will relax the condition of monotonicity on the extreme coefficients and thereby obtain the following results which in turn generalize many known results besides the above results.

III. MAIN RESULTS

Theorem 1:If $p(q) = \sum_{l=0}^{n} q^l a_l$ is a polynomial of degree n with quaternionic coefficients and quaternionic variable where $a_l = \alpha + \beta i + \gamma j + \delta k, 0 \leq l \leq n$ and satisfying

$$\alpha_r \geq \alpha_{r-1} \geq \ldots \geq \alpha_{s+1} \geq \alpha_s, \beta_r \geq \beta_{r-1} \geq \ldots \geq \beta_{s+1} \geq \beta_s,$$
$$\gamma_r \geq \gamma_{r-1} \geq \ldots \geq \gamma_{s+1} \geq \gamma_s, \delta_r \geq \delta_{r-1} \geq \ldots \geq \delta_{s+1} \geq \delta_s,$$

$0 \leq s < r \leq n$, then all the zeros $p(q)$ lie in

$$|q| \leq \frac{1}{|a_n|}\left[M_r + N_\alpha + N_\beta + N_\gamma + M_\delta\right]$$

Where $N_t = |t_0| - t_s + t_r$,

$$M_r = \sum_{i=r+1}^{n}\left(\Delta\alpha_i + \Delta\beta_i + \Delta\gamma_i + \Delta\delta_i\right),$$

$$M_s = \sum_{i=1}^{s}\left(\Delta\alpha_i + \Delta\beta_i + \Delta\gamma_i + \Delta\delta_i\right)$$

$$\Delta x_i = |x_i - x_{i-1}|.$$

Remark: If we take r = n then Theorem 1 reduces to the result due to Tripathi [9, Theorem 3.1].

Theorem 2: If $p(q) = \sum_{l=0}^{n} q^l a_l$ is a polynomial of degree n with quaternionic coefficients and quaternionic variable $a_l = \alpha + \beta i + \gamma j + \delta k, 0 \leq l \leq n$ such that for a integer r and a non-negative integer s, $0 \leq s < r \leq n$, we have $a_r \geq a_{r-1} \geq \ldots \geq a_{s+1} \geq a_s$, then $p(q)$ does not vanish in

$$|q| < \min\left(1, \frac{|a_0|}{V_r + a_r - a_s + |a_n| + V_s}\right)$$

Where $V_r = \sum_{i=r+1}^{n} \Delta\alpha_i$ and $V_s = \sum_{i=1}^{s} \Delta\alpha_i$.

Remark : Applying Theorem 2 to the polynomials $p(q)$ with real coefficients ,i.e., $\beta = \gamma = \delta = 0$, we get The result due to shah et al. [8] and if we put $r = n$ in Theorem 2 , we get the result due to Tripathi [9 ,Theorem 3.7].

IV. LEMMAS

To prove the results stated above, we need the following lemma due to Gentili and Stoppato [3].

Lemma 1. Let $f(q) = \sum_{l=0}^{n} q^l a_l$ and $g(q) = \sum_{l=0}^{n} q^l b_l$ be two quaternionic power series with radii of convergence greater than R.The regular product of $f(q)$ and $g(q)$ is defined as $(f * g)(q) = \sum_{l=0}^{n} q^l c_l$, where $c_l = \sum_{t=0}^{n} a_t b_{l-t}$.

Let $|q_0|$ < R, then $(f * g)(q_0) = 0$ if and only if $f(q_0) = 0$ or $f(q_0) \neq 0$ implies $g\left(f(q)^{-1} q_0 f(q)\right) = 0$.

V. PROOF OF THE THEOREMS

Proof of Theorem 1. Consider the polynomial

$$f(q) = \sum q^t (a_t - a_{t-1}) + a_0$$

and

$p(q) * (1 - q) = f(q) - q^{n+1} a_n$. Therefore by Lemma 1, $p(q) * (1 - q) = 0$ if and only if either $p(q) = 0$ or $p(q) \neq 0$ Implies $p(q)^{-1} q p(q) - 1 = 0$ .If $p(q) \neq 0$, then $q = 1$. Therefore, the only zeros of $p(q) * (1 - q)$ are $q = 1$ and the zeros of $p(q)$. Thus for $|q| = 1$, we get

$$|f(q)| \leq |a_0| + \sum_{t=1}^{n} |a_t - a_{t-1}|$$
$$\leq |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0|$$
$$+ \sum_{t=1}^{n}\left(\Delta\alpha_i + \Delta\beta_i + \Delta\gamma_i + \Delta\delta_i\right)$$

$$= |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0|$$
$$+ \sum_{t=r+1}^{n} (\Delta\alpha_i + \Delta\beta_i + \Delta\gamma_i + \Delta\delta_i)$$
$$+ \sum_{t=s+1}^{r} (\Delta\alpha_i + \Delta\beta_i + \Delta\gamma_i + \Delta\delta_i)$$
$$+ \sum_{t=1}^{s} (\Delta\alpha_i + \Delta\beta_i + \Delta\gamma_i + \Delta\delta_i)$$

Now by using the hypothesis, we get for $|q| = 1$

$$|f(q)| \leq M_r + |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + T + M_s$$

where

$$T = (\alpha_r - \alpha_s) + (\beta_r - \beta_s) + (\gamma_r - \gamma_s) + (\delta_r - \delta_s)$$

Since

$$\max_{|q|=1} \left| q^n * f\left(\frac{1}{q}\right) \right| = \max_{|q|=1} \left| f\left(\frac{1}{q}\right) \right| = \max_{|q|=1} |f(q)|$$

, therefore $q^n * f\left(\frac{1}{q}\right)$ has the same bound on $|q| = 1$.

Thus for $|q| = 1$

$$\left| q^n * f\left(\frac{1}{q}\right) \right| \leq M_r + |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + T + M_s.$$

Applying maximum modulus theorem [3] for quarternionic polynomials, it follows that for | $|q| \leq 1$

$$\left| q^n * f\left(\frac{1}{q}\right) \right|$$
$$\leq M_r + |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + T + M_s.$$

That is for | $|q| \leq 1$

$$\left| f\left(\frac{1}{q}\right) \right| \leq \frac{1}{|q|^n} [M_r + |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + T + M_s]$$

Replacing q by $\frac{1}{q}$, we get for $|q| > 1$

$$|f(q)| \leq [M_r + |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + T + M_s] |q|^n$$

But $|p(q) * (1-q)| = |f(q) - q^{n+1} a_n|$

$$\geq |a_n| |q|^{n+1} - |f(q)|.$$

Therefore we have for $|q| \geq 1$

$$|p(q) * (1-q)|$$
$$\geq [|a_n||q| - (M_r + |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + T + M_s)]$$

This implies that $|p(q) * (1-q)| > 0$, i.e. $p(q) * (1-q) \neq 0$ if

$$|q| > \frac{1}{|a_n|} [M_r + |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + T + M_s].$$

Since the only zeros of $p(q) * (1-q)$ are $q = 1$ and the zeros of $p(q)$. Therefore, $p(q) \neq 0$ for

$$|q| > \frac{1}{|a_n|} [M_r + |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + T + M_s].$$

Hence all the zeros of $p(q)$ lie in

$$|q| \leq \frac{1}{|a_n|} [M_r + N_\alpha + N_\beta + N_\gamma + M_\delta]$$

This proves Theorem 1.

Proof of Theorem 2. Define the reciprocal polynomial

$$W(q) = q^n * p\left(\frac{1}{q}\right) = \sum_{t=0}^{n} q^{n-t} a_t.$$

Let $W(q) * (1-q) = g(q) - q^{n+1} a_0$, where

$$g(q) = \sum_{t=1}^{n} q^{n-t+1} (a_{t-1} - a_t) + a_n.$$

Thus for $|q| = 1$, we get

$$|g(q)| \leq \sum_{t=1}^{n} |a_{t-1} - a_t| + |a_n|$$

Now by using the hypothesis and continuing as in Theorem 1, it follows that for $|q| > 1$

$$|g(q)| \leq [V_r + a_r - a_s + |a_n| + V_s] |q|^n.$$

But

$$|W(q) * (1-q)| = |g(q) - q^{n+1} a_0|$$
$$\geq |a_0| |q|^{n+1} - |g(q)|$$
$$\geq [|a_0||q| - (V_r + a_r - a_s + |a_n| + V_s)] |q|^n,$$

If

$$|q| > \frac{V_r + a_r - a_s + |a_n| + V_s}{|a_0|},$$

that is $W(q) * (1-q) \neq 0$ for

$$|q| > \frac{V_r + a_r - a_s + |a_n| + V_s}{|a_0|}.$$

Hence all the zeros of $W(q) * (1-q)$ whose modulus greater than 1 lie in

$$|q| \le \frac{V_r + a_r - a_s + |a_n| + V_s}{|a_0|}.$$

Therefore, all the zeros of $W(q)$ lie in

$$|q| \ge \min\left(1, \frac{|a_0|}{V_r + a_r - a_s + |a_n| + V_s}\right).$$

Hence the polynomial does not vanish in

$$|q| < \min\left(1, \frac{|a_0|}{V_r + a_r - a_s + |a_n| + V_s}\right).$$

That proves Theorem 2.

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